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HEAT TRANSFER IN MAGNETOHYDRODYNAMIC FLOW BETWEEN PARALLEL PLATES*

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Abstract—An analysis is presented of convective heat transfer in the fully developed laminar flow of an incompressible conducting fluid between parallel plates through a transverse magnetic field. In particular the earlier analysis of Siegel involving non-conducting plates is corrected and extended to plates of finite conductivity.

Résumé—Cet article présente une étude sur la transmission de chaleur par convection dans un écoulement laminaire pleinement établi d'un fluide conducteur incompressible, entre des plaques parallèles à travers un champ magnétique. En particulier, l'étude la plus récente de Siegel est corrigée et étendue à des plaques de conductivité finie.

Zusammenfassung—Es wird der konvektive Wärmeübergang in einer inkompressiblen elektrisch leitenden Flüssigkeit untersucht für voll ausgebildete Laminarströmung zwischen parallelen Platten und einem dazwischen angelegten Magnetfeld. Insbesondere wird die frühere Analyse von Siegel für nichtleitende Platten verbessert und auf elektrisch leitende Platten ausgedehnt.

Аннотация—В статье проводится анализ процесса конвективного переноса тепла при установившемся ламинарном движении несжимаемой электропроводящей жидкости между параллельными пластинами через поперечное магнитное поле. Уточняется метод Сигела для непроводящих пластин и осуществляется перенос его на случай движения между пластинами с конечной проводимостью.

NOMENCLATURE

- a, thermal diffusivity;
- A, mean temperature gradient;
- B, magnetic induction;
- c, velocity of light;
- C_p, fluid specific heat at constant pressure;
- E, electric field intensity;
- H, magnetic field intensity;
- h, thickness of channel wall;
- j, current density;
- k, thermal conductivity;
- λ, magnetic viscosity;
- L, half-height of channel;
- μ, magnetic permeability;
- M, Hartmann number;
- ν, kinematic viscosity;
- p, pressure;
- P, pressure gradient;
- Pr, Prandtl number;
- q, heat flux;
- ρ, density;
- Re, hydrodynamic Reynolds number;
- Re_m, magnetic Reynolds number;
- σ, electrical conductivity;
- u, fluid velocity;
- $\bar{u}$, mean fluid velocity;
- u_0, cosh M/(cosh M-1) times centerline velocity;
- W, ratio of electrical resistances of channel walls and fluid;
- x, co-ordinate in flow direction;
- y, co-ordinate transverse to flow and magnetic field;
- z, co-ordinate in magnetic field direction.

Subscripts

- p, confining walls or plates;
- f, fluid.

INTRODUCTION

RECENTLY an analysis was presented by Siegel [1] of the convective heat transfer for fully

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developed laminar flow in a parallel plate channel with an imposed uniform wall heat flux, in the situation that the fluid is incompressible and electrically conducting, the plates are electrical insulators, and there is an impressed magnetic field transverse to the flow. The magnetic field flattens the parabolic velocity distribution usual to such a flow so that one then finds relatively higher fluid velocities near the wall as in turbulent flow [2]. As a consequence of this and of ohmic heating arising from induced currents in the fluid which provide an internal heat source, the temperature difference between wall and fluid is altered.

The problem becomes considerably more complex if one considers the plates to be electrically conducting. Now the fluid is not in a net-current-zero state, the magnitude and distribution of current in the fluid depends on the relative resistance of fluid and plates, and there is ohmic heating in the parallel plates as well as in the fluid. One can consider the plates to be electrically conducting without providing heat to the fluid by replacing them conceptually with plates of infinite conductivity connected in a coplanar sense (to preserve one-dimensionality) through an external load resistance [2, 3]. The situation is of some interest in that it provides a very crude model to a portion of the heat transfer problem in magnetohydrodynamic power-generating ducts of large aspect ratio, in the case when the walls normal to the applied field are nominally insulators but do in fact conduct under operating conditions.

It is the purpose of this note to extend Siegel's analysis to this case of plates of arbitrary electrical conductivity. For convenience and the sake of completeness some of the background is summarized. The results are presented only formally for the reason that they are complex and in practice one would doubtless resort to machine computation in such heat transfer problems in order to take into account more of the physical variables.

FLOW DESCRIPTION

Consider the steady laminar flow of a viscous incompressible fluid of constant conductivity σ_f through a parallel plate channel of height $2L$. (c.g.s. units are used throughout this paper.) A uniform magnetic field $B = B_z$ is applied. The walls parallel to the field are at $y = \pm\infty$. Continuity requires that the velocity $u = u_x(z)$ be uniform in x, with $u_y = u_z = 0$. Then one can show [3, 4] the fully developed velocity profile to be

$$u = u_0\left(1 - \frac{\cosh Mz/L}{\cosh M}\right) \tag{1}$$

where u_0, which is $\cosh M/(\cosh M - 1)$ times the fluid velocity on the channel center line, is given by

$$u_0 = \frac{Pc^2}{\sigma_f B_z^2}\frac{M(1+W)}{MW + \tanh M}, \tag{2}$$

the quantity $P = -\partial p/\partial x$ is the uniform pressure gradient in the channel, M is the Hartmann number, defined as $M = (B_z L/c)\,[\sigma_f/(\nu\rho)]^{1/2}$, with ν and ρ the kinematic viscosity and density, respectively, and W is the ratio of the electrical resistance of the fluid to that of the plates. Thus, with plates of equal thickness h and uniform conductivity σ_p, one may write

$$W = \frac{\sigma_p \,.\, 2h}{\sigma_f \,.\, 2L}. \tag{3}$$

If one is dealing not with parallel plates connected at $y = \pm\infty$ but with coplanar connection through an external load, the quantity $(2h\sigma_p)^{-1}$ should include the series resistance of the external load per unit channel length. The average fluid velocity in the channel is found from equations (1) and (2) to be

$$\bar{u} = u_0\left(1 - \frac{1}{M}\tanh M\right) = \frac{Pc^2}{\sigma_f B_z^2}\,\frac{(1+W)(M - \tanh M)}{MW + \tanh M}. \tag{4}$$

With the assumption that the induced magnetic field in the z-direction is small compared to the applied field, one obtains from the usual magnetohydrodynamic equations and boundary conditions [5] the following expression for the induced fields, current distributions and pressure gradient:

$$(E_x)_f = (E_x)_p = (E_z)_f = (E_z)_p = 0, \tag{5}$$

$$(E_y)_f = (E_y)_p = \frac{\bar{u}B_z}{c(1+W)}, \tag{6}$$

$$(\mu H_y)_f = (\mu H_y)_p = 0, \qquad (7)$$

$$(\mu H_x)_f = \frac{Re_m B_z}{M}\left[\frac{\sinh M_z/L}{\cosh M} - \frac{z}{L}\,\frac{MW + \tanh M}{1 + W}\right], \qquad (8)$$

$$(\mu H_x)_p = \frac{4\pi\mu_p\sigma_p}{c^2}\cdot\frac{\bar{u}B_z}{1 + W}\cdot z, \qquad (9)$$

$$(j_y)_f = \frac{\sigma_f B_z}{c}\left(\frac{\bar{u}}{1 + W} - u\right), \qquad (10)$$

$$(j_y)_p = \frac{\sigma_p \bar{u} B_z}{c(1 + W)}, \qquad (11)$$

and

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{\sigma_f B_z^2}{\rho c^2}\left[u_0 - \frac{\bar{u}}{1 + W}\right], \qquad (12)$$

in which μ is permeability, $Re_m = 2Lu_0/\lambda$ is the magnetic Reynolds number, and $\lambda = c^2/(4\pi\mu_f\sigma_f)$ is the fluid magnetic viscosity. Implicit in equations (5) through (9) is conservation of current within any element of channel length, viz.

$$\int_{-L}^{L}(j_y)_f \mathrm{d}z + 2h(j_y)_p = 0.$$

Ohmic heating in the fluid and in the plates is computed as j^2/σ from equations (10) and (11).

HEAT TRANSFER ANALYSIS

Suppose that the externally imposed heat flux q through the confining walls is uniform with x, consistent with the fluid velocity being independent of x. If one neglects viscous dissipation compared to the transverse heat transport and, further, disregards any temperature effects due to ohmic losses in the plates, then the temperature gradient, A, in the channel can be written

$$A = \frac{\partial T_f}{\partial x} = \frac{\partial \bar{T}_f}{\partial x} = \frac{1}{L\bar{u}\rho C_p}\left[g + \int_0^L \frac{(j_y)_f^2}{\sigma_f}\,\mathrm{d}z\right], \qquad (13)$$

where $\bar{T}_f$ is the mean fluid temperature and C_p is the fluid specific heat at constant pressure. It follows that one can take

$$T_f(x,z) = Ax + G(z). \qquad (14)$$

The energy balance for the system can be written as

$$u\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial z^2} + \frac{1}{\rho C_p}\frac{(j_y)_f^2}{\sigma_f}, \qquad (15)$$

or, using equation (14), as

$$uA = \alpha\frac{\mathrm{d}^2 G}{\mathrm{d}z^2} + \frac{1}{\rho C_p}\frac{(j_y)_f^2}{\sigma_f}, \qquad (16)$$

with boundary conditions consistent with equation (14) being

$$G = 0,\ z = \pm L; \quad \mathrm{d}G/\mathrm{d}z = 0,\ z = 0. \qquad (17)$$

The choice $G = 0$ rather than $G =$ constant is purely for convenience. The quantity α is the thermal diffusivity which for present purposes is defined by

$$\alpha = k/(\rho C_p), \qquad (18)$$

where k is the thermal conductivity. (The validity of this definition for α must be examined in any real calculation, since replacing k by $\alpha\rho C_p$ in developing these results requires that ρC_p be constant in the fluid). Integration of equation (16) yields

$$G(z) = C_1\left(\frac{z^2}{L^2} - 1\right) + C_2\left(\frac{\cosh Mz/L}{\cosh M} - 1\right) + C_3\left(\frac{\cosh 2Mz/L}{\cosh 2M} - 1\right), \qquad (19)$$

where

$$C_1 = \frac{u_0 A L^2}{2\alpha} - \frac{u_0^2 Pr M^2}{2C_p}\left[1 + \frac{\operatorname{sech}^2 M}{2} - \frac{2}{1 + W}\frac{\bar{u}}{u_0} + \left(\frac{1}{1 + W}\frac{\bar{u}}{u_0}\right)^2\right], \qquad (20)$$

$$C_2 = -\frac{u_0 A L^2}{\alpha M^2} + \frac{2u_0^2 Pr}{C_p}\left[1 - \frac{\bar{u}}{u_0(1 + W)}\right], \qquad (21)$$

and

$$C_3 = -\frac{u_0^2 Pr}{8C_p}\cosh 2M \operatorname{sech}^2 M, \qquad (22)$$

in which $Pr = \nu/\alpha$ is the Prandtl number, and the quantity A, the mean temperature gradient defined in equation (13), can be written

$$A = (L\bar{u}\rho C_p)^{-1}\left\{q + P^2L^3/(\rho\nu) \times\left[\frac{1}{M^2} - \frac{2(1 + W)\tanh M}{M^2(MW + \tanh M)} + \frac{(1 + W)^2}{2M}\,\frac{\sinh M\cosh M + M}{\cosh^2 M(MW + \tanh M)^2}\right]\right\}. \qquad (23)$$

From equation (14), together with the obvious definition of mean fluid temperature, one may write the difference between wall and mean fluid temperature as

$$T_w - \bar{T}_f = Ax - \frac{1}{2L\bar{u}}\int_{-L}^{L} T_f(x,z)u(z)\mathrm{d}z$$
$$= -\frac{1}{2L\bar{u}}\int_{-L}^{L} G(z)u(z)\mathrm{d}z. \qquad (24)$$

Carrying out the indicated integrations yields

$$T_w - \bar{T}_f = \left(\frac{2}{3}\frac{u_0}{\bar{u}} - \frac{2}{M^2}\right) C_1 + \left(\frac{3}{2} - \frac{u_0}{2\bar{u}}\tanh^2 M\right) C_2 + \left(1 - \frac{u_0}{\bar{u}}\frac{\tanh 2M \operatorname{sech}^2 M}{6M}\right) C_3, \qquad (25)$$

where the coefficients C_1, C_2, and C_3 are as defined in equations (20–22). Inserting these coefficients and rearranging terms makes it possible to write equation (25) as

$$T_w - \bar{T}_f = \frac{qL}{k}\,\phi(M) + \frac{(RePr)^2\nu\alpha}{L^2C_p}\,\chi(M,W), \qquad (26)$$

in which the Reynolds number is defined as $\bar{u}L/\nu$, and the function $\phi(M)$ can be written as

$$\phi(M) = \frac{1}{2M}\left(\frac{u_0}{\bar{u}}\right)^2\left[\frac{2M}{3} + \frac{1}{M}\left(\tanh^2 M - 5\frac{\bar{u}}{u_0}\right)\right]. \qquad (27)$$

One can show $\phi(M)$ to be the same as that defined by Siegel. The function $\chi(M, W)$ is given by

$$\chi(M,W) = \frac{1}{2}\left(\frac{u_0}{\bar{u}}\right)^2\left\{\frac{u_0}{\bar{u}}\left(\frac{MW + \tanh M}{1+W}\right)^2\left[\frac{u_0}{\bar{u}}\left(\frac{2}{3} + \frac{\tanh^2 M}{M^2}\right) - \frac{5}{M^2}\right]\left[1 - \frac{2(1+W)\tanh M}{MW + \tanh M} + \frac{(1+W)^2 M}{2}\,\frac{\sinh M\cosh M + M}{\cosh^2 M(MW + \tanh M)^2}\right] - \left(\frac{2}{3}\frac{u_0}{\bar{u}} - \frac{2}{M^2}\right) M^2\left[1 + \frac{\operatorname{sech}^2 M}{2} - \frac{2}{1+W}\frac{\bar{u}}{u_0} + \left(\frac{1}{1+W}\frac{\bar{u}}{u_0}\right)^2\right] + 2\left(3 - \frac{u_0}{\bar{u}}\tanh^2 M\right)\left[1 - \frac{\bar{u}}{u_0(1+W)}\right] - \frac{1}{4}\left(1 - \frac{\bar{u}}{u_0}\,\frac{\tanh 2M\tanh^2 M}{6M}\right) \times \cosh 2M \operatorname{sech}^2 M\right\}. \qquad (28)$$

Siegel's function $\chi(M)$, which should be the same as $\chi(M, 0)$ evaluated from equation (28), is not correct; the error in his paper apparently occurs in the steps between obtaining $G(z)$ and $T_w - \bar{T}_f$.

To illustrate the interplay of the quantities q, M, and W it is instructive to obtain an approximation for $T_w - \bar{T}_f$ with W arbitrary and M large (it appears that M may be of the order of 100 in typical magnetohydrodynamic power ducts). One finds for $\phi(M)$ the following

$$\phi(M) \cong \frac{1}{3} + \frac{2}{3M} - \frac{1}{M^2}, \qquad (30)$$

and for $\chi(M, W)$

$$\chi(M,W) \cong \frac{1}{2}\frac{M}{(1+W)^2} \times\left[-\frac{1}{3}(1+W)^2 + \frac{1}{M}\left(-\frac{11}{6} + \frac{W}{3} + \frac{W^2}{6}\right) + \frac{1}{M^2}\left(-\frac{27}{12} + \frac{3W}{2} + \frac{17W^2}{12}\right)\right]. \qquad (31)$$

For M sufficiently large, one finds that $\chi(M,W)$ becomes independent of W and equation (26) can be written

$$T_w - \bar{T}_f \cong \frac{1}{3}\frac{qL}{k} - \frac{(RePr)^2\nu\alpha}{6L^2C_p}M. \qquad (32)$$

To illustrate the magnitude of $T_w - \bar{T}_f$, one can rewrite equation (32) for mercury, viz.,

$$T_w - \bar{T}_f \cong 22qL - \tfrac{1}{8}\bar{u}^2M, \qquad (33)$$

with q in cal/cm^2 s, L in cm, and $\bar{u}$ in cm/s. It is interesting to note that equation (33) indicates a possibility of sign change in $T_w - \bar{T}_f$ for a given q as M is increased. This is physically reasonable when one notes from equations (13) and (14) that for $M = 0$, with q as defined, the temperature difference corresponds to heat conduction into the fluid. If, on the other hand the magnetic field becomes high enough so that ohmic heating in the fluid exceeds the energy supplied to the fluid through q, the temperature difference should change sign.

Consider now the case when the term in χ is negligible compared to that in ϕ, as for example would be the case for a very low velocity flow through a strong magnetic field. Then one finds, as did Siegel, that $T_w - \bar{T}_f$ is reduced from its value with no field. In the case of M large, the decrease is from $(17/35)qL/k$ to $(1/3)qL/k$, or nearly 50 per cent.

One final comment with respect to equation (31) is that the value $W = 1$ corresponds to the optimum operating point of idealized magnetohydrodynamic ducts, i.e. $W = 1$ corresponds to equality of external load and internal generator resistances.

DISCUSSION

The foregoing is in some respects an exercise, insofar as its applicability to heat transfer calculations in real magnetohydrodynamic flows is concerned. Working fluids are generally compressible, conductivities are temperature and density dependent, uniform wall heat fluxes are only a convenient fiction, flows are in fact three-dimensional (side wall effects [4, 6] with or without power extraction) and also radiative heat transfer will play a role. Most important however, is the high probability that flows of real interest will be turbulent [7]. In power-generating devices the walls to which this parallel plate analysis refers will be nominally insulators when cool, but will conduct in operation as a result of property changes at high temperatures and possible reaction with seeding materials added to the working fluid to render it conducting. Moreover this conduction may be in addition to that provided by side wall electrodes. It would appear on the basis of preliminary studies made of magnetohydrodynamic power-generating devices that in practice one will be dealing with M large and $W \neq 0$.

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